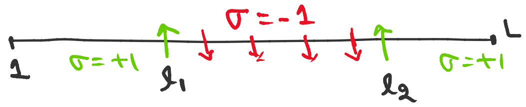
Landau and peierls argument

Thursday, May 13, 2021 4:33 PM

Consider a 1-d Ising model at zero field

$$H = -\sum_{ij} J_{ij}\,\sigma_i \sigma_j \qquad \text{say} \quad J_{ij} = \frac{J}{|i-j|^{\alpha}}$$

Landue argument:



Entropy for a pair of domain wall $\simeq k_B 2 \log L$ for large L

Energy cost for creating a pair of domain walls

$$= 2 \sum_{j=l_1+1}^{l_2-1} \left[\sum_{i=1}^{l_1} J_{ij} + \sum_{i=l_2}^{L} J_{ij} \right] \qquad \text{with } x = j/L,\ y = i/L$$

$$= 2 J L^{2-\alpha} \int_{x_1+1/L}^{x_2-1/L} dx \left\{ \int_{1/L}^{x_1} dy + \int_{x_2}^{1} dy \right\} \frac{1}{|x-y|^{\alpha}}$$

For $\alpha > 2$: Energy cost is finite in large L limit $\Rightarrow$ Entropy wins $\Rightarrow$ no long range order.

For $\alpha < 2$: Energy cost grows faster than $\log L$

$\Rightarrow$ Energy wins $\Rightarrow$ long range order.

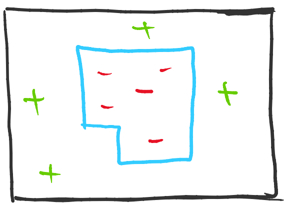
For $\alpha = 2$:

$$\Delta E = 2J \int_{x_1+1/L}^{x_2-1/L} dx \left\{ \int_{1/L}^{x_1} dy + \int_{x_2}^{1} dy \right\} \frac{1}{(x-y)^2} \simeq 2J \log L$$

Therefore a phase transition at a finite temperature

Remark: This problem of Ising model with $1/r^2$ potential in 1-d is related to Kondo effect [Anderson], also motivated Kosterlitz and Thouless for their now famous work on KT transition.

In 2-d nearest neighbor Ising model:



For a domain wall of length l

Energy cost $= 2 J l$

Number of possible domain boundaries of length l

$\Omega(l) \sim N \cdot (q-1)^{l} \Rightarrow$ Entropy $\simeq k_B \log \Omega$

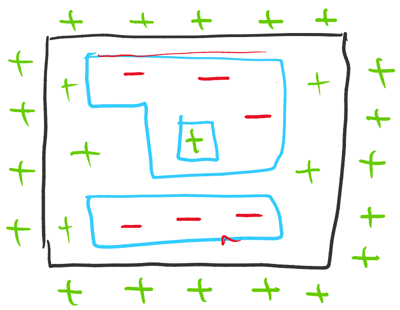
$= l \log(q-1) + \dots$

Therefor Energy cost $\simeq$ Entropy gain

$\Rightarrow$ It is possible to have phase transition.

This argument was made more rigorous by Pierels and Griffiths, giving first convincing evidence that 2d Ising model undergoes a phase transition.

Pierels - Griffiths argument for 2d Ising model:



Consider zero field Ising model on a square lattice.

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \quad \text{nearest neighbor interactions.}$$

Question: given a boundary condition that all spins at the boundary are held fixed, is the bulk magnetization non-zero at a finite β?

Solution: Let N_- be the number of down spins in a config with the positive boundary condition. In such a configuration, there will be many domain walls of different length. Total number of sites inside a domain of boundary length l is bounded by $\frac{l^2}{4}$.

Let $n(l)$:= number of ways (shapes) of drawing a closed domain wall of length l.

Then

$$\langle N_- \rangle \leq \sum_{l=4,6,8,\dots} \frac{l^2}{4} \sum_{k=1}^{n(l)} \langle \nu_k(l) \rangle$$

number of domain wall of k-th shape of length l.

Then bulk magnetization

$$m = \frac{\langle N_+ \rangle - \langle N_- \rangle}{N} = 1 - 2\frac{\langle N_- \rangle}{N} \quad \text{is nonzero, if } \frac{\langle N_- \rangle}{N} < \frac{1}{2}.$$

How to get $\langle \nu_k(l) \rangle$?

$$\langle \nu_k(l) \rangle \leq e^{-2\beta J l} \cdot N$$

similarly $\quad n(l) \leq 3^{l-1} \quad$ for square lattice.

Together

$$\langle N_- \rangle \leq \sum_{l=4,6,8} \frac{l^2}{4} \cdot N \cdot 3^{l-1} \cdot e^{-2\beta J l}$$

$$\Rightarrow \frac{\langle N_- \rangle}{N} \leq \frac{1}{48} \sum_{l=4,6,}^{\infty} l^2 \left(3 e^{-2\beta J}\right)^l \quad \text{in thermodynamic limit}$$

$$\leq \frac{1}{2} \quad \text{for } \beta J \geq \beta_c J = 0.717$$

This means, for $\beta > \beta_c$, magnetization $m > 0$. Therefore, certainly there is nonzero spontaneous magnetization for $T < T_c$ $\left(k_B T_c = \frac{1}{\beta_c}\right)$. Ofcourse, the exact T_c is lower with $J\beta_c^{exact} = 0.4406$.